

Cut locus structures on graphs

Jin-ichi Itoh and Costin Vîlcu

January 14, 2013

Abstract. Motivated by a fundamental geometrical object, the cut locus, we introduce and study a new combinatorial structure on graphs.

Math. Subj. Classification (2000): 57R70, 53C20, 05C10

1 Introduction

The motivation of this work comes from a basic notion in riemannian geometry, that we shortly present in the following. In this paper, by a *surface* we always mean a complete, compact and connected 2-dimensional riemannian manifold without boundary.

The *cut locus* $C(x)$ of the point x in the surface S is the set of all extremities (different from x) of maximal (with respect to inclusion) shortest paths (geodesic segments) starting at x ; for basic properties and equivalent definitions refer, for example, to [13] or [16]. The notion was introduced by H. Poincaré [15] and gain, since then, an important place in global riemannian geometry.

For surfaces S is known that $C(x)$, if not a single point, is a local tree (i.e., each of its points z has a neighbourhood V in S such that the component $K_z(V)$ of z in $C(x) \cap V$ is a tree), even a tree if S is homeomorphic to the sphere. A *tree* is a set T any two points of which can be joined by a unique Jordan arc included in T .

All our graphs are finite, connected, undirected, and may have multiple edges or loops.

S. B. Myers [14] established that the cut locus of a real analytic surface is (homeomorphic to) a graph, and M. Buchner [3] extended the result for manifolds of arbitrary dimension. For not analytic riemannian metrics on S ,

cut loci may be quite large sets, see the work of J. Hebda [6] and of the first author [9]. Other contributions to the study of this notion were brought, among others, by M. Buchner [2], [4], H. Gluck and D. Singer [5], J. Hebda [7], J. Itoh [8], K. Shiohama and M. Tanaka [17], T. Zamfirescu [18], [19], A. D. Weinstein [20].

We show in another paper [10] that *for every graph G there exists a surface S_G and a point x in S whose cut locus $C(x)$ is isomorphic to G ; rephrasing, every graph can be realized as a cut locus.*

If G has an odd number q of generating cycles then any surface S_G realizing G is non-orientable, but if q is even then one cannot generally distinguish, by simply looking to the graph G , whether S_G is orientable or not: explicit examples show that both possibilities can occur [11]. In other words, seen as a graph, *the cut locus does not encode the orientability of the ambient space.*

This is our main motivation to endow graphs with a combinatorial structure – that of cut locus structure, or shortly CL-structure.

In this paper we treat combinatorial aspects of this new notion: in Section 2 we introduce and discuss this notion, in Section 3 we give two planar representations of CL-structures, and in the last section we enumerate all such structures on “small” graphs.

In a second paper [10] we show that every CL-structure actually corresponds to a cut locus on a surface, while in a subsequent one [11] we consider the orientability of the surfaces realizing CL-structures as cut loci. In particular, any graph endowed with a CL-structure does encode the orientability of the ambient space where it lives as a cut locus. An upper bound on the number of CL-structures on a graph is given in [12].

At the end of this section we recall a few notions from graph theory, in order to fix the notation.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by B the set of all *bridges* in the graph G ; i.e., edges whose removal disconnects G . Each non-vertex component of $G \setminus B$ is called a *2-connected component* of G .

A *k-graph* is a graph all vertices of which have degree k .

The power set \mathcal{E} of E becomes a Z_2 -vector space over the two-element field Z_2 if endowed with the symmetric difference as addition. \mathcal{E} can be thought of as the space of all functions $E \rightarrow Z_2$, and called the (binary) *edge space* of G . The (binary) *cycle space* is the subspace \mathcal{Q} of \mathcal{E} generated by

(the edge sets of) all simple cycles of G . If G is seen as a simplicial complex, \mathcal{Q} is the space of 1-cycles of G with mod 2 coefficients.

2 Cut locus structures

Definition 2.1 *A G -patch on the graph G is a topological surface P_G with boundary, containing (a graph isomorphic to) G and contractible to it.*

Remark 2.2 *Every boundary component of a patch is homeomorphic to a circle, as a 1-dimensional manifold without boundary.*

Definition 2.3 *A G -strip (or a strip on G , or simply a strip, if the graph is clear from the context), is a G -patch with 1-component boundary; i.e., whose boundary is one topological circle; see Figure 1 (a).*

The next remark gives the geometrical background for the notion of cut locus structure.

Remark 2.4 *Consider a point x on a surface S , and a geodesic segment $\gamma : [0, l] \rightarrow S$ parameterized by arclength, with $\gamma(0) = x$ and $\gamma(l) \in C(x)$. For $\varepsilon > 0$ smaller than the injectivity radius at x , and hence smaller than l , the point $\gamma(l - \varepsilon)$ is well defined. Since $S \setminus C(x)$ is contractible to x along geodesic segments, and thus homeomorphic to an open disk, the union over all γ s of those points $\gamma(l - \varepsilon)$ is homeomorphic to the unit circle, and therefore the set $\bigcup_{\gamma} \{\gamma(l - \mu) : 0 \leq \mu \leq \varepsilon\}$ is a $C(x)$ -strip.*

Definition 2.5 *A cut locus structure (shortly, a CL-structure) on the graph G is a strip on the cyclic part G^{cp} of G .*

Remark 2.6 *We show in another paper [10], with geometrical tools, the converse to Remark 2.4: every CL-structure can be obtained (with some suitable surface and point on the surface) as described in Remark 2.4.*

Remark 2.7 *Each G -strip defines a circular order around each vertex of G , and thus a rotation system. Conversely, one can alternatively define a G -strip as the graph associated to a rotation system, together with a 2-cell embedding having precisely one face. We choose not to follow this way, and to keep in our presentation as much as possible of the geometrical intuition.*

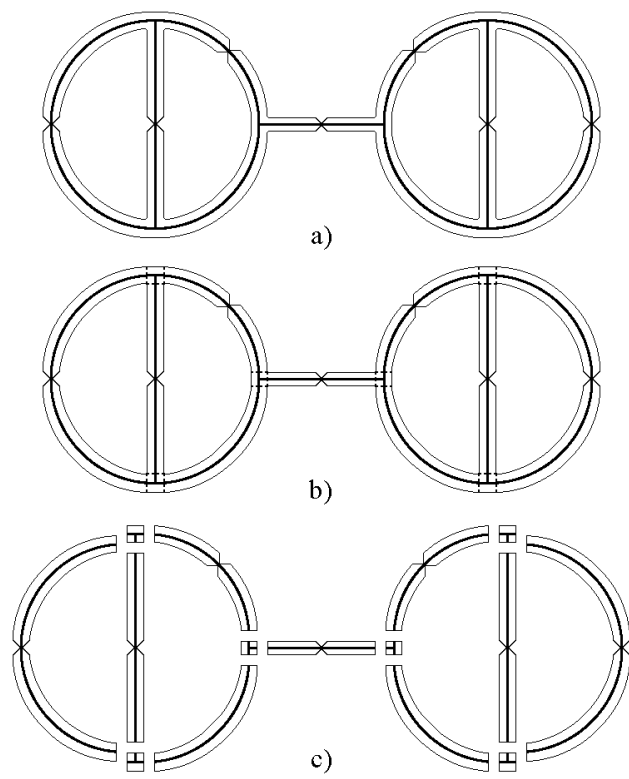


Figure 1: A strip and its elementary decomposition.

Definition 2.8 *An elementary strip is an edge-strip (arc-strip) or a point-strip; i.e., a strip defined by the graph with precisely one edge (arc) of different extremities, respectively by the graph consisting of one single vertex.*

Definition 2.9 *An elementary decomposition of a G -patch P_G is a decomposition of P_G into elementary strips such that:*

- *each edge-strip corresponds to precisely one edge of G ;*
- *each point-strip corresponds to precisely one vertex of G ; see Figure 1 (b) and (c).*

Remark 2.10 *Our notion of “ G -patch” is equivalent to that of “fibered surface” introduced by M. Bestvina and M. Handel: “a fibered surface is a compact surface F with boundary which is decomposed into arcs and into polygons that are modeled on k -junctions, $k = 1, 2, 3, \dots$. The components of the subsurface fibered by arcs are strips. Shrinking the decomposition elements to points produces a graph G , where vertices (of valence k) correspond to (k -) junctions and strips to edges. We can think of G as being embedded in F , representing the spine of F ” [1]. We choose the most (in our opinion) appropriate name for our purpose, and thus different from theirs.*

In order to easier handle a CL-structure, we associate to it an object of combinatorial nature. To this goal, denote by \mathcal{P} and \mathcal{A} the set of all point-strips, respectively edge-strips, of a CL-structure \mathcal{C} on the graph G .

Definition 2.11 *Consider an elementary decomposition of the G -strip P_G such that each elementary strip has a distinguished face, labeled $\bar{0}$. The face opposite to the distinguished face will be labeled $\bar{1}$. Here, $\bar{0}$ and $\bar{1}$ are the elements of the 2-element group (\mathbb{Z}_2, \oplus) .*

To each pair $(v, e) \in V \times E$ consisting of a vertex v and an edge e incident to v , we associate the \mathbb{Z}_2 -sum $\bar{s}(v, e)$ of the labels of the elementary strips $\nu \in \mathcal{P}$, $\varepsilon \in \mathcal{A}$ associated to v and e ; i.e., $\bar{s}(v, e) = \bar{0}$ if the distinguished faces of ν and ε agree to each other, and $\bar{1}$ otherwise. Therefore, to any cut locus structure \mathcal{C} we can associate a function $s_{\mathcal{C}} : E \rightarrow \{\bar{0}, \bar{1}\}$,

$$s_{\mathcal{C}}(e) = \bar{s}(v, e) \oplus \bar{s}(v', e), \quad (1)$$

where v and v' are the vertices of the edge $e \in E$.

We call the function $s_{\mathcal{C}}$ defined by (1) the companion function of \mathcal{C} .

The value $s_{\mathcal{C}}(e)$ above can be thought of as the switch of the edge e .

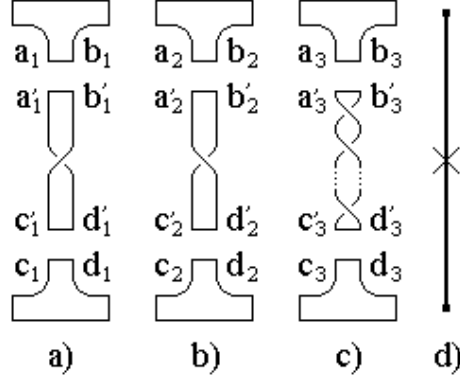


Figure 2: Equivalent CL-structures (a), (b) and (c), and schematic representation (d). The edge-strip at (a) corresponds to a rectangular band whose base is π -rotated “to the left” with respect to the top; the edge-strip at (b) corresponds to a rectangular band whose base is π -rotated “to the right” with respect to the top; the edge-strip at (c) corresponds to a rectangular band whose base is $(2k + 1)\pi$ -rotated “to the left” with respect to the top.

Definition 2.12 Assume first that the graph G is 2-connected. Two CL-structures $\mathcal{C}, \mathcal{C}'$ on G are called equivalent if their companion functions are equivalent: i.e., $s_{\mathcal{C}}$ and $s_{\mathcal{C}'}$ are equal, up to a simultaneous change of the distinguished face for all elementary strips in G (i.e., either $s_{\mathcal{C}} = s_{\mathcal{C}'}$, or $s_{\mathcal{C}} = \bar{1} \oplus s_{\mathcal{C}'}$).

If G is not 2-connected, the CL-structures $\mathcal{C}, \mathcal{C}'$ on G are called equivalent if their companion functions are equivalent on every 2-connected component of G . See Figure 2.

Definition 2.13 An edge-strip P_e (or simply an edge e) in a CL-structure \mathcal{C} is called switched if $s_{\mathcal{C}}(e) = \bar{1}$.

Proposition 2.14 If two CL-structures on the same graph G are equivalent then the corresponding G -strips are homeomorphic surfaces.

Proof: We may assume that G is cyclic.

Assume, moreover, that we have two CL-structures on G , whose companion functions are equivalent on every 2-connected component of G . The desired homeomorphism can be constructed inductively, extending it with each new “gluing” of an elementary strip, see Figure 2. \square

3 Representations of CL-structures

We propose two ways to planary represent a CL-structure \mathcal{C} on the graph G .

Definition 3.1 *The graph representation of \mathcal{C} starts with some planar representation of G , and afterward points out the CL-structure, see Figure 3 (a).*

Definition 3.2 *The natural representation of \mathcal{C} starts by representing in the plane each vertex-strip such that its distinguished face is “up”, and afterward connects the vertex-strips by edge-strips. The idea is illustrated by Figure 3 (b) and (c).*

Remark 3.3 *Consider the natural representation of a CL-structure on a cubic graph. We shall overwrite an “x” to the drawn image of an edge if its strip is switched, and an “=” to the drawn image of an edge if its strip is not-switched. See Figures 4 and 3.*

Remark 3.4 *Neither the natural representation, nor the graph representation, of a CL-structure on a graph is unique.*

Proposition 3.5 *For any planar cubic graph G and any CL-structure on G , the natural representation and the graph representation coincide, up to planar homeomorphisms.*

Proof: This follows from the definitions above. □

Example 3.6 *If the 3-graph G is not planar, Proposition 3.5 is not true. An easy example, obtained from a flat torus of rectangular fundamental domain (see the procedure described in Remark 2.4), is illustrated by Figure 5.*

Directly from the definitions we have the following.

Lemma 3.7 *In any natural representation of a strip, each cycle-patch contains at least one switched edge-strip.*

We can give four open questions.

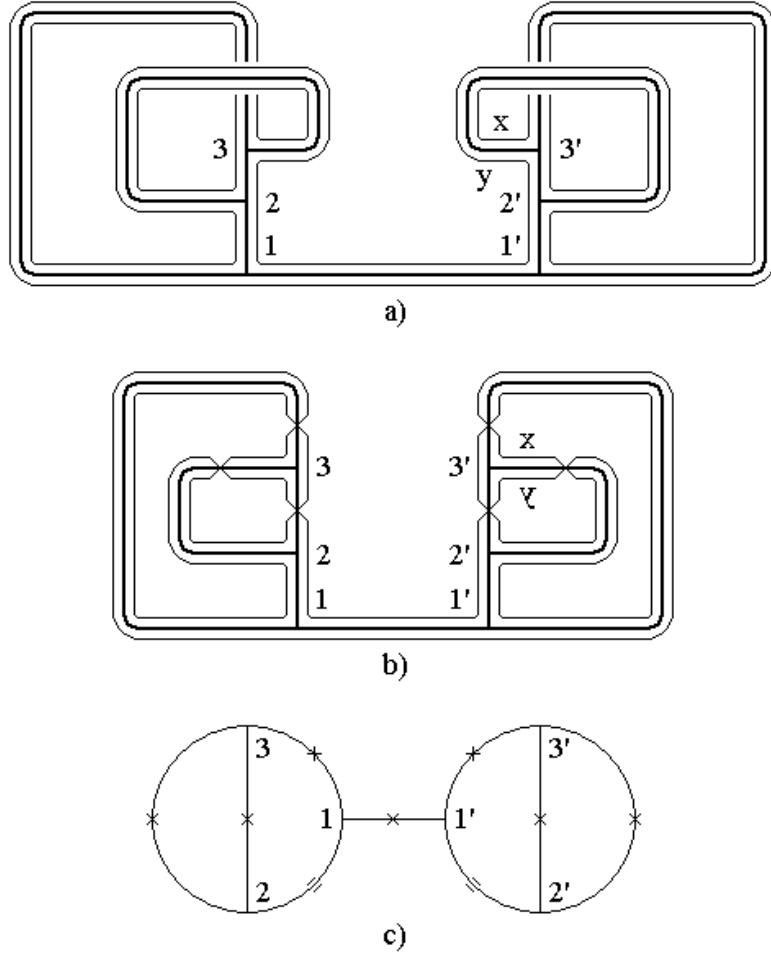


Figure 3: Representations of CL-structures. a) *Graph representation* of a strip. b) Intermediate step to obtain (c). c) *Natural representation* for the strip at (a). Additional points x, y are indicated to make clear the transformation.

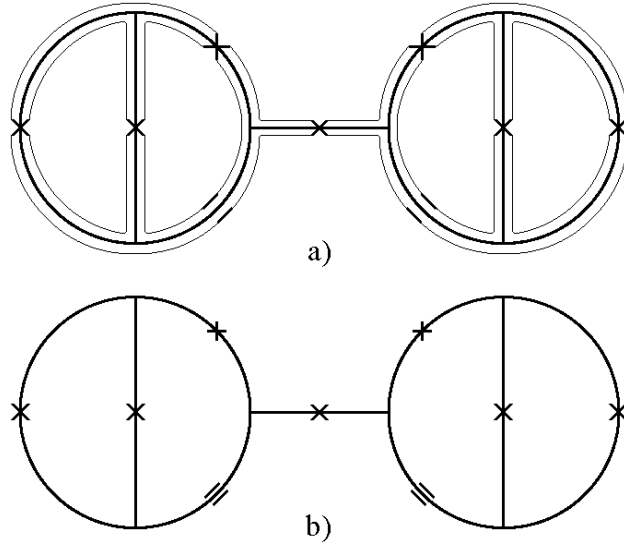


Figure 4: Schematic representation of the strip in Figure 1 (a).

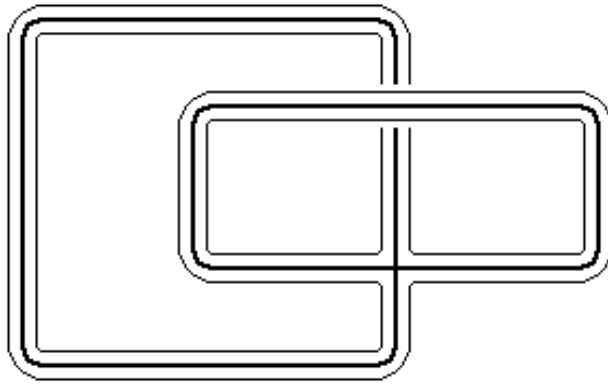


Figure 5: CL-structure obtained from a flat torus of rectangular fundamental domain.

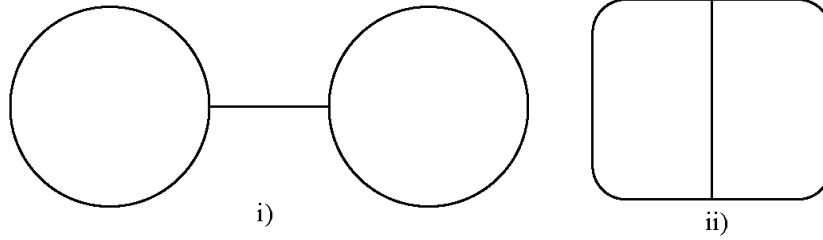


Figure 6: All 3-graphs with 2 generating cycles.

Question 3.8 *Characterize the companion functions of CL-structures in the set $\mathcal{S} = \{s : E \rightarrow \{\bar{0}, \bar{1}\}\}$.*

Question 3.9 *A planar graph is, by definition, a graph which can be represented in the plane without crossings (self-intersections). As we have seen in Example 3.6, there are CL-structures on (not cubic) planar graphs whose natural representations in the plane necessarily produce crossings. What is the minimal number of such crossings which guarantees a planar natural representation?*

The same question can be asked for non planar graphs too, where the (minimal number of necessary) crossings of the graphs is a new parameter.

Question 3.10 *How many CL-structures can coexist on the same graph?*

Some (not sharp) upper bound will be given in [11].

Question 3.11 *Which of the graphs with q generating cycles has the largest number of different CL-structures?*

We shall address in the following section the last two questions above, for graphs with two and three generating cycles.

4 CL-structures on small graphs

We present in this section all distinct cut locus structures on 3-graphs with $q = 2, 3$ generating cycles.

The following statement can be obtained by straightforward inductive constructions.

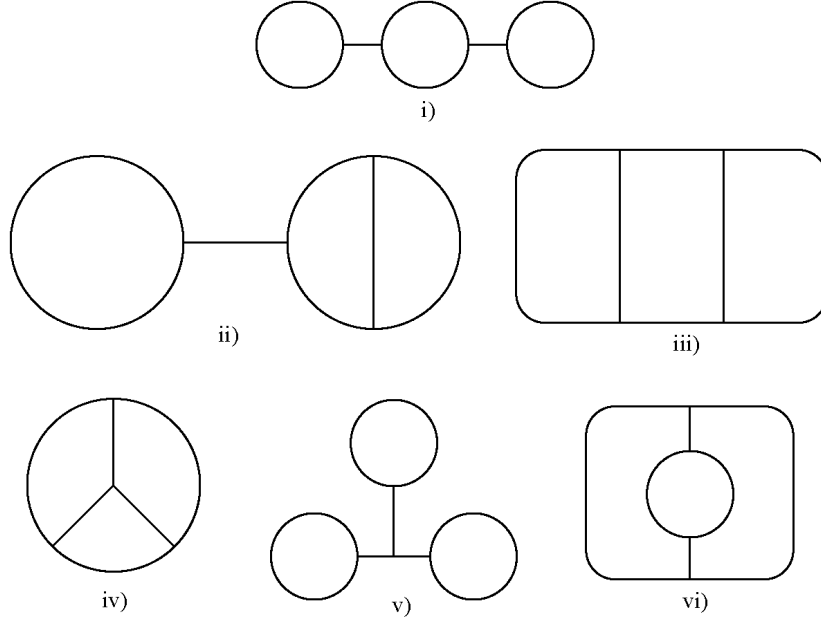


Figure 7: All 3-graphs with 3 generating cycles.

Lemma 4.1 *There are precisely 2, respectively 6, distinct 3-graphs with 2, respectively 3, generating cycles, see Figures 6 and 7.*

Theorem 4.2 a) *There are precisely 3 non-equivalent CL-structures on the 3-graphs with 2 generating cycles, see Figures 8 and 9.*

b) *There are precisely 17 non-equivalent CL-structures on the 3-graphs with 3 generating cycles, see Figures 10 – 15.*

Proof: We employ the natural representation of CL-structures. It is straightforward to generate all patches on the graphs in Figures 6 – 7, to keep only

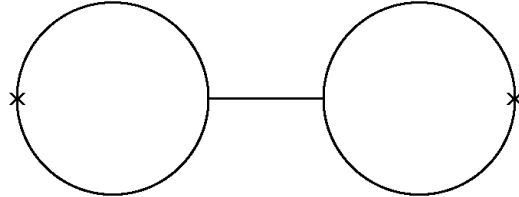


Figure 8: Unique CL-structure on the graph in Figure 6 (i).

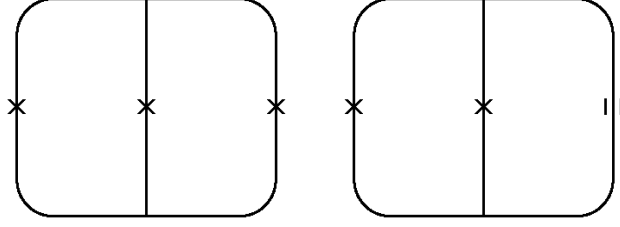


Figure 9: 2 CL-structures on the graph in Figure 6 (ii).

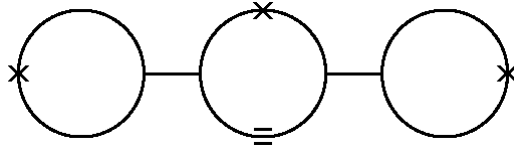


Figure 10: Unique CL-structure on the graph in Figure 7 (i).

the strips (by the use of Lemma 3.7), and to use Definition 2.12 and the symmetries of the graphs to identify equivalent CL-structures. \square

Our last result shows that the case of CL-structures on 3-graphs is, in some sense, sufficient. For, define the *degree of a graph* as the maximal degree of its vertices.

Theorem 4.3 *Any CL-structures on a graph with q generating cycles and degree larger than 3 can be obtained from CL-structures on 3-graphs with q generating cycles, by contracting non-switched edge-strips.*

Proof: Fix q ; we consider only graphs with q generating cycles, and proceed by induction over the number of vertices of degree larger than 3. Denote by $D(G)$ this number for the graph G .

Assume the cyclic graph G has $D(G) \geq 1$, and choose a vertex v in G with $\deg(v) = d > 3$.

Let \mathcal{C} be a CL-structure on G , and denote by v_1, \dots, v_d the neighbours of v in G , and by T the subtree of G rooted at v , with leaves v_1, \dots, v_d . Let G^- be the complement of T in G , and \mathcal{C}^- be the union of patches naturally induced by \mathcal{C} on G^- . Let $s_{\mathcal{C}}^-$ be the restriction of the companion function $s_{\mathcal{C}}$ of \mathcal{C} to G^- .

Replace T in G by a tree T_3 of leaves v_1, \dots, v_d , all of whose internal vertices have degree 3 (T_3 is generally not unique), and denote by G^v the

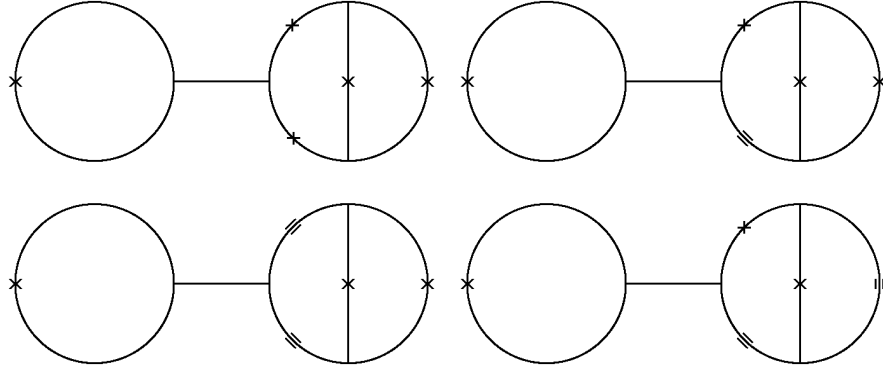


Figure 11: 4 CL-structures on the graph in Figure 7 (ii).

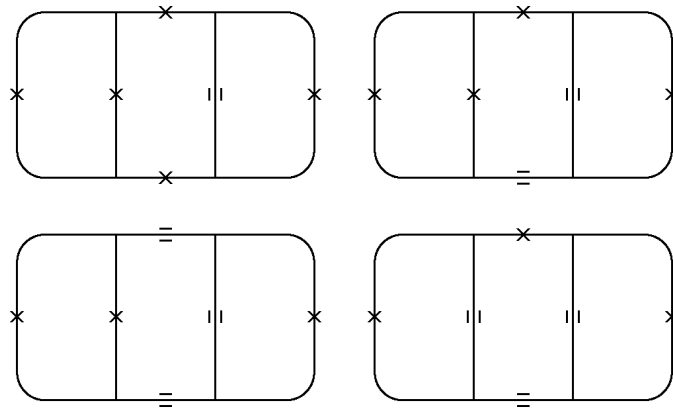


Figure 12: 4 CL-structures on the graph in Figure 7 (iii).

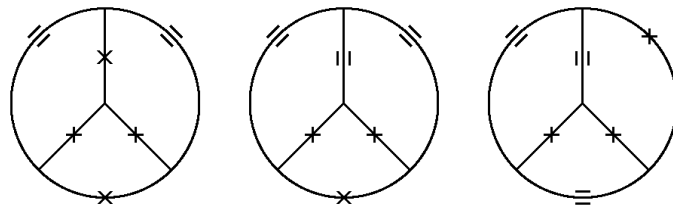


Figure 13: 3 CL-structures on the graph in Figure 7 (iv).

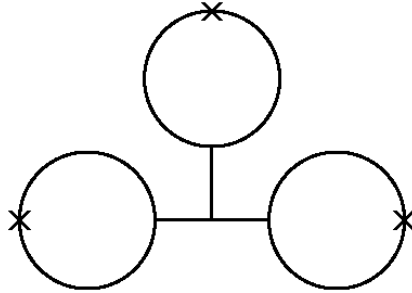


Figure 14: Unique CL-structure on the graph in Figure 7 (v).

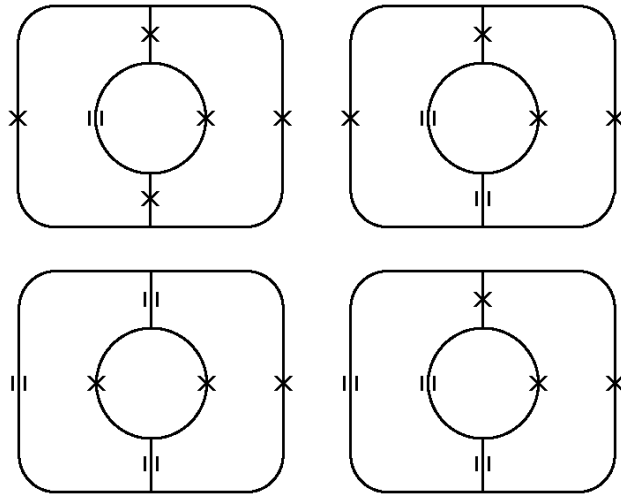


Figure 15: 4 CL-structures on the graph in Figure 7 (vi).

new graph. Now complete \mathcal{C}^- to a CL-structure \mathcal{C}^v on G^v , by extending $s_{\mathcal{C}}^-$ on the internal edges of T_3 with $\bar{0}$, and on the external edges of T_3 with the original values of $s_{\mathcal{C}}$. Observe that \mathcal{C}^v is indeed a CL-structure on G^v , and $D(G^v) = D(G) - 1$, so the proof is complete. \square

Acknowledgement C. Vîlcu was partially supported by the grant PN II Idei 1187 of the Romanian Government.

References

- [1] M. Bestvina and M. Handel *Train-tracks for surface homeomorphisms*, Topology **34** (1995), 109-140
- [2] M. A. Buchner, *Stability of the cut locus in dimensions less than or equal to 6*, Invent. Math. **43** (1977), 199-231
- [3] M. A. Buchner, *Simplicial structure of the real analytic cut locus*, Proc. Amer. Math. Soc. **64** (1977), 118-121
- [4] M. A. Buchner, *The structure of the cut locus in dimension less than or equal to six*, Compos. Math. **37** (1978), 103-119
- [5] H. Gluck and D. Singer, *Scattering of geodesic fields I and II*, Ann. Math. **108** (1978), 347-372, and **110** (1979), 205-225
- [6] J. Hebda, *Metric structure of cut loci in surfaces and Ambrose's problem*, J. Differ. Geom. **40** (1994), 621-642
- [7] J. Hebda, *Cut loci of submanifolds in space forms and in the geometries of Möbius and Lie*, Geom. Dedicata **55** (1995), 75-93
- [8] J. Itoh, *Some considerations on the cut locus of a riemannian manifold*, in *Geometry of geodesics and related topics (Tokyo, 1982)*, 29-46, Adv. Stud. Pure Math., 3, North-Holland, Amsterdam, 1984
- [9] J. Itoh, *The length of a cut locus on a surface and Ambrose's problem*, J. Differ. Geom. **43** (1996), 642-651
- [10] J. Itoh and C. Vîlcu, *Every graph is a cut locus*, to appear
- [11] J. Itoh and C. Vîlcu, *Orientable realizations of graphs as a cut loci*, to appear
- [12] J. Itoh and C. Vîlcu, *On the number of cut locus structures on graphs*, to appear

- [13] S. Kobayashi, *On conjugate and cut loci*, Global differential geometry, MAA Stud. Math. **27** (1989), 140-169
- [14] S. B. Myers, *Connections between differential geometry and topology I and II*, Duke Math. J. **1** (1935), 376-391, and **2** (1936), 95-102
- [15] H. Poincaré, *Sur les lignes géodésiques des surfaces convexes*, Trans. Amer. Math. Soc. **6** (1905), 237-274
- [16] T. Sakai, *Riemannian Geometry*, Translation of Mathematical Monographs 149, Amer. Math. Soc. 1996
- [17] K. Shiohama and M. Tanaka, *Cut loci and distance spheres on Alexandrov surfaces*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, 1996, 531-559
- [18] T. Zamfirescu, *Many endpoints and few interior points of geodesics*, Invent. Math. **69** (1982) 253-257
- [19] T. Zamfirescu, *On the cut locus in Alexandrov spaces and applications to convex surfaces*, Pac. J. Math. **217** (2004) 375-386
- [20] A. D. Weinstein, *The cut locus and conjugate locus of a riemannian manifold*, Ann. Math. **87** (1968), 29-41

Jin-ichi Itoh

Faculty of Education, Kumamoto University
 Kumamoto 860-8555, Japan
 j-ito@ipo.kumamoto-u.ac.jp

Costin Vilcu

Institute of Mathematics “Simion Stoilow” of the Romanian Academy
 P.O. Box 1-764, Bucharest 014700, Romania
 Costin.Vilcu@imar.ro